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Description of anharmonic effects with generalized coherent states

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Abstract

The Marumori–Yamamura–Tokunga boson expansion is used to describe multiboson processes. The reliability of the time evolution predicted by generalized coherent states has been investigated numerically by comparing its dynamics with the exact one. A link between the generalized coherent states and deformed bosons is established.

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1. Introduction

A mapping, defined in the framework of the Marumori–Yamamura–Tokunga (MYT) boson expansion [1], from a 'physical' boson space onto a 'model' boson space, was proposed in [2] to describe multiphonon processes, such as multiphonon absorption or emission. In particular, a multiboson coherent state adequate to be used as a trial function in the time-dependent variational method, was introduced. The new coherent state has the form of the extended coherent states discussed in [3], and in fact can be interpreted in terms of the deformed boson scheme [2, 4–7]. The same generalized coherent states have been applied to the Lipkin model [8].

In the present paper we test the MYT boson mapping and the proposed generalized coherent states by studying the time evolution of a system given in terms of a schematic model appropriate to describe the coupling between different kinds of bosons. In particular, we are interested in describing the decay of a collective state into a multiboson state. We will test the original Glauber coherent state, the generalized coherent state introduced in [2], as well as two other coherent states which, by construction, possess a conserved quantity of the system, namely the number of particles: the $SU(2)$ coherent [9] and a generalized version.

We will show that the new bosons obtained within the MYT scheme are equivalent to the introduction of deformed bosons. Recently, the MYT boson expansion generated by

q-deformed boson operators which obey the $quon$-algebra has been applied to study different Hamiltonians [6].

This paper is organized as follows: in section 2 the Hamiltoninan in the new boson space is calculated using the MYT boson expansion method, then the classical analogue is calculated using Glauber coherent states, the generalized coherent states are introduced in section 4, afterwards we show that the formalism is equivalent to using deformed bosons and finally we present some numerical results and draw some conclusions.

2. The Hamiltonian in the new space

Sometimes, in physical systems, a highly anharmonic coupling between distinct degrees of freedom occurs and is responsible for remarkable effects. For instance, the excitation of a giant resonance in a heavy nucleus may induce its fission. The mechanism responsible for such a process is obviously related to energy transfer from the giant resonance degree of freedom to the fission degree of freedom. In order to better understand similar physical mechanisms it may be interesting to investigate the dynamics determined by Hamiltonians of the form

$$\hat{H} = \omega\hat{c}^\dagger\hat{c} + \Omega\hat{b}^\dagger\hat{b} + g(\hat{c}^\dagger\hat{b}^n + \hat{b}^{\dagger n}\hat{c}) \tag{1}$$

where $\hat{c}$ and $\hat{b}$ are boson operators: $[\hat{c}, \hat{c}^\dagger] = 1$, $[\hat{b}, \hat{b}^\dagger] = 1$, $[\hat{c}, \hat{b}] = 0$, etc. We assume that $\omega = n\Omega$. Since $[(\omega\hat{c}^\dagger\hat{c} + \Omega\hat{b}^\dagger\hat{b}), \hat{c}^\dagger\hat{b}^n] = (\omega - n\Omega)\hat{c}^\dagger\hat{b}^n$, this condition ensures that the unperturbed part of the Hamiltonian commutes with the perturbation, being a conserved quantity. The coupling between both degrees of freedom has most dramatic effects, because the energy is more freely exchanged between them.

Let $|0)$ be the vacuum associated with the boson operators $\hat{b}^\dagger, \hat{b}$. The subspace spanned by the kets $\frac{1}{\sqrt{(kn)!l!}}\hat{b}^{\dagger kn}\hat{c}^{\dagger l}|0)$, $k, l = 0, 1, \ldots, \infty$ is left invariant by the Hamiltonian (1). In this subspace, the operator $\hat{b}^n$ behaves, essentially, as a single boson operator d. The MYT boson expansion was originally devised as a method to consistently replace a fermion pair by a single boson, but its basic idea can be applied in the present case. The MYT boson expansion is a mapping from vectors and operators referring to some physical system (in our case the bosons b) into vectors and operators referring to the auxiliary system (the bosons d), under the important requirement that operator matrix elements are preserved. This method allows us to express the operator $\hat{b}^{\dagger n}, \hat{b}^n$ in terms of the operators $\hat{d}^\dagger, \hat{d}$. We denote by $|0\rangle$ the vacuum associated with $\hat{d}^\dagger, \hat{d}$. The ket $\frac{1}{\sqrt{(kn)!}}\hat{b}^{\dagger kn}|0)$ is mapped into the ket $\frac{1}{\sqrt{k!}}\hat{d}^{\dagger k}|0\rangle$. The image of the operator $b^{\dagger n}$ is an operator $\hat{d}^\dagger f(\hat{d}^\dagger\hat{d})$ which has the same matrix elements between corresponding states. This requirement leads to the condition

$$(0|\frac{1}{\sqrt{((k+1)n)!}}b^{(k+1)n}(b^{\dagger n})\frac{1}{\sqrt{(kn)!}}b^{\dagger kn}|0) = \langle 0|\frac{1}{\sqrt{(k+1)!}}\hat{d}^{k+1}(\hat{d}^\dagger)f(\hat{d}^\dagger\hat{d})\frac{1}{\sqrt{k!}}\hat{d}^{\dagger k}|0\rangle$$

so that

$$\frac{((k+1)n)!}{\sqrt{((k+1)n)!}\sqrt{(kn)!}} = \frac{(k+1)!f(k)}{\sqrt{(k+1)!}\sqrt{k!}}.$$

Thus

$$f(k) = \frac{\sqrt{(kn+1)\cdots(kn+n)}}{\sqrt{k+1}}.$$

Let

$$F_n(k) = \frac{f^2(k)}{n!} = \prod_{p=1}^{n-1}\left(1 + \frac{n}{p}k\right) = \frac{(n(k+1)-1)!}{(nk)!(n-1)!}. \tag{2}$$

The desired boson images of $\hat{b}^{\dagger n}$ and $\hat{b}^n$ are, respectively, $\hat{d}^\dagger\sqrt{n!F_n(\hat{N})}$ and $\sqrt{n!F_n(\hat{N})}\hat{d}$, where $\hat{N} = \hat{d}^\dagger\hat{d}$. Since

$$(0|\frac{1}{\sqrt{(kn)!}}\hat{b}^{kn}(\hat{b}^\dagger\hat{b})\frac{1}{\sqrt{(kn)!}}\hat{b}^{\dagger kn}|0) = n\langle 0|\frac{1}{\sqrt{k!}}\hat{d}^k(\hat{d}^\dagger\hat{d})\frac{1}{\sqrt{k!}}\hat{d}^{\dagger k}|0\rangle$$

the boson image of $\hat{b}^\dagger\hat{b}$ is $n\hat{d}^\dagger\hat{d}$. Summarizing, within the present framework we obtain the following images:

$$\hat{b}^{\dagger n} \rightarrow \hat{d}^\dagger\sqrt{n!F_n(\hat{N})} \qquad \hat{b}^n \rightarrow \sqrt{n!F_n(\hat{N})}\hat{d} \qquad \hat{b}^\dagger\hat{b} \rightarrow n\hat{d}^\dagger\hat{d}. \tag{3}$$

The Hamiltonian (1) is replaced by

$$\hat{H} = \omega(\hat{c}^\dagger\hat{c} + \hat{d}^\dagger\hat{d}) + G\left(\hat{c}^\dagger\sqrt{F_n(\hat{N})}\hat{d} + \hat{d}^\dagger\sqrt{F_n(\hat{N})}\hat{c}\right) \tag{4}$$

where $G = \sqrt{n!}g$. $\hat{H}$ can be written as a sum of two parts which commute

$$\hat{H} = 2\omega\hat{\sigma} + G\hat{C} \qquad 2\hat{\sigma} = \hat{c}^\dagger\hat{c} + \hat{d}^\dagger\hat{d} \qquad \hat{C} = \hat{c}^\dagger\sqrt{F_n(\hat{N})}\hat{d} + \hat{d}^\dagger\sqrt{F_n(\hat{N})}\hat{c}.$$

In fact we have

$$[\hat{H}, \hat{\sigma}] = [\hat{H}, \hat{C}] = [\hat{C}, \hat{\sigma}] = 0.$$

3. The standard coherent state and the classical analogue

It is well known that the MYT boson expansion is the quantal analogue of a canonical transformation. This analogy is very clear in the present example. The classical analogue of $\hat{H}$ is the Hamiltonian

$$H(\gamma^*, \beta^*, \gamma, \beta) = \omega\gamma^*\gamma + \Omega\beta^*\beta + g(\gamma^*\beta^n + \beta^{*n}\gamma)$$

where $(\gamma, \mathrm{i}\gamma^*)$ and $(\beta, \mathrm{i}\beta^*)$ are canonically conjugate pairs: $\mathrm{i}\{\gamma, \gamma^*\} = 1, \mathrm{i}\{\beta, \beta^*\} = 1, \{\gamma, \beta\} = 0$, etc, and $\omega = n\Omega$. We observe that $H(\gamma^*, \beta^*, \gamma, \beta)$ is the expectation value of $\hat{H}$ in the coherent state $|\gamma, \beta\rangle = \exp(\gamma\hat{c}^\dagger + \beta\hat{b}^\dagger)|0\rangle$, where $|0\rangle$ is the boson vacuum

$$H(\gamma^*, \beta^*, \gamma, \beta) = \frac{\langle\gamma, \beta|\hat{H}|\gamma, \beta\rangle}{\langle\gamma, \beta|\gamma, \beta\rangle}$$
$$\langle\gamma, \beta|\gamma, \beta\rangle = \exp(\gamma^*\gamma + \beta^*\beta).$$

In the following, by coherent state 1 (CSI) we mean the coherent state $|C\rangle_1 = |\gamma, \beta\rangle$. In order to obtain simple dynamical equations for our system, some transformations are now introduced. The transformation

$$\beta^n = n^{n/2}\delta(\delta^*\delta)^{(n-1)/2} \qquad \beta^{n*} = n^{n/2}\delta^*(\delta^*\delta)^{(n-1)/2}$$

is canonical and leads to the replacement of $H(\gamma^*, \beta^*, \gamma, \beta)$ by

$$H(\gamma^*, \delta^*, \gamma, \delta) = \omega(\gamma^*\gamma + \delta^*\delta) + n^{n/2}g(\gamma^*\delta + \delta^*\gamma)(\delta^*\delta)^{(n-1)/2}.$$

The subsequent replacement $\gamma = \sqrt{p}\,\mathrm{e}^{\mathrm{i}\phi}$, $\delta = \sqrt{P}\,\mathrm{e}^{\mathrm{i}\Phi}$ leads to

$$H(p, P, \phi, \Phi) = \omega(p + P) + 2n^{n/2}g\cos(\phi - \Phi)\sqrt{pP^n}. \tag{5}$$

From the equations of motion it follows that

$$\dot{p} + \dot{p} = 0 \qquad \dot{\phi} - \dot{\Phi} = 2n^{n/2}g\cos(\phi - \Phi)\frac{\mathrm{d}}{\mathrm{d}p}\sqrt{pP^n}.$$

The first equation is not surprising since $n\gamma^*\gamma + \beta^*\beta$ is a constant of motion. In the last equation, we have taken into account the fact that P only depends on p, with $\mathrm{d}P/\mathrm{d}p = -1$. Finally we obtain

$$\ddot{p} = 2n^n g^2 \frac{\mathrm{d}}{\mathrm{d}p}(pP^n) \qquad P = p_0 - p. \tag{6}$$

Let us now write, formally

$$\hat{c}^\dagger = \mathrm{e}^{\mathrm{i}\hat{\phi}}\sqrt{\hat{p}} \qquad \hat{c} = \sqrt{\hat{p}}\,\mathrm{e}^{-\mathrm{i}\hat{\phi}} \qquad \hat{d}^\dagger = \mathrm{e}^{\mathrm{i}\hat{\Phi}}\sqrt{\hat{P}} \qquad \hat{d} = \sqrt{\hat{P}}\,\mathrm{e}^{-\mathrm{i}\hat{\Phi}}.$$

Then, $[\hat{\Phi}, \hat{P}] = \mathrm{i}$ implies $[\hat{d}, \hat{d}^\dagger] = 1$ and $[\hat{\phi}, \hat{p}] = \mathrm{i}$ implies $[\hat{c}, \hat{c}^\dagger] = 1$. In terms of $\hat{\Phi}, \hat{P}, \hat{\phi}, \hat{p}, \hat{H}$, equation (4) becomes

$$\hat{H} = \omega(\hat{p} + \hat{P}) + G\left[\mathrm{e}^{\mathrm{i}\hat{\phi}}\sqrt{\hat{p}F_n(\hat{P})\hat{P}}\,\mathrm{e}^{-\mathrm{i}\hat{\Phi}} + \mathrm{e}^{\mathrm{i}\hat{\Phi}}\sqrt{\hat{P}F_n(\hat{P})\hat{p}}\,\mathrm{e}^{-\mathrm{i}\hat{\phi}}\right].$$

In this form, $\hat{H}$ is analogous to $H(p, P, \phi, \Phi)$, (5).

4. The generalized coherent state

We introduce the generalized coherent state

$$|C\rangle = |V, v\rangle = \mathcal{N}\exp[V\hat{d}^\dagger(F_n(\hat{N}))^{-1/2} + v\hat{c}^\dagger]|0\rangle \qquad (\hat{d}|0\rangle = \hat{c}|0\rangle = 0) \tag{7}$$

where V, V^*, v, v^* are variational parameters. The generalized coherent state is an eigenvector of $\sqrt{F_n(\hat{N})}\hat{d}$. It may be easily shown that

$$\sqrt{F_n(\hat{N})}\hat{d}|C\rangle = V|C\rangle. \tag{8}$$

In the following, by coherent state 2 (CSII) we mean the coherent state $|C\rangle_2 = |V, v\rangle$ defined by equation (7). The normalization constant $\mathcal{N}$, which ensures that $\langle C|C\rangle = 1$, is easily computed. We find

$$\begin{aligned}\mathcal{N}^{-2} &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(V^*V)^k(v^*v)^l}{(k!)^2F_n(0)\cdots F_n(k-1)(l!)^2}\langle 0|\hat{d}^k\hat{c}^l\hat{c}^{l\dagger}\hat{d}^{k\dagger}|0\rangle \\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(V^*V)^k(v^*v)^l}{k!F_n(0)\cdots F_n(k-1)l!} \\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(n!)^k}{(nk)!l!}(V^*V)^k(v^*v)^l\end{aligned}$$

where the obvious relation

$$F_n(0)\cdots F_n(k) = \frac{(n(k+1))!}{(k+1)!(n!)^{k+1}}$$

has been used. It is convenient to introduce the functions

$$\mathcal{Z}_n(V^*V) = \sum_k \frac{(n!)^k}{(nk)!}(V^*V)^k$$

$$\mathcal{G}_n(V^*V) = \frac{\sum_k (n!)^k k(V^*V)^{k-1}((nk)!)^{-1}}{\sum_k (n!)^k(V^*V)^k((nk)!)^{-1}} = \frac{\partial \log \mathcal{Z}_n}{\partial(V^*V)}$$

which are useful in the following discussion. For instance, the normalization constant is given by

$$\mathcal{N}^{-2} = \mathcal{Z}_n(V^*V)\,\mathrm{e}^{v^*v}.$$

For $n = 1$ we recover the usual Glauber coherent state normalization, $\mathcal{Z}_1 = \exp(V^*V)$, while for $n = 2$ we have $\mathcal{Z}_2 = \cosh\sqrt{2V^*V}$. We compute now the Lagrangian which determines the time evolution of the complex variables V, v,

$$\mathcal{L} = \frac{\mathrm{i}}{2}(\langle C|\dot{C}\rangle - \langle\dot{C}|C\rangle) - \langle C|\hat{H}|C\rangle \tag{9}$$

where $|\dot{C}\rangle = \partial_t|C\rangle$, $\langle\dot{C}| = \partial_t\langle C|$. We observe that

$$\partial_t(\mathcal{N}^{-1}|C\rangle) = [\dot{V}\hat{d}^\dagger(F_n(\hat{N}))^{-1/2} + \dot{v}\hat{c}^\dagger]\exp[V\hat{d}^\dagger(F_n(\hat{N}))^{-1/2} + v\hat{c}^\dagger]|0\rangle.$$

Thus,

$$\begin{aligned}\frac{1}{\mathcal{N}}\langle C|\partial_t\left(\frac{1}{\mathcal{N}}|C\rangle\right) &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left(\frac{\dot{V}v}{(k-1)!l!} + \frac{V\dot{v}}{k!(l-1)!}\right)\frac{V^*v^*(V^*V)^{k-1}(v^*v)^{l-1}}{k!F_n(0)\cdots F_n(k-1)l!}\langle 0|\hat{d}^k\hat{c}^l\hat{c}^{l\dagger}\hat{d}^{k\dagger}|0\rangle \\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(k\dot{V}v + lV\dot{v})\frac{V^*v^*(V^*V)^{k-1}(v^*v)^{l-1}}{k!F_n(0)\cdots F_n(k-1)l!} \\ &= \left(V^*\dot{V}\frac{\partial\mathcal{Z}_n}{\partial(V^*V)} + v^*\dot{v}\mathcal{Z}_n\right)\mathrm{e}^{v^*v}\end{aligned}$$

and

$$\langle C|\dot{C}\rangle - \langle\dot{C}|C\rangle = (V^*\dot{V} - \dot{V}^*V)\mathcal{Z}_n^{-1}\frac{\partial\mathcal{Z}_n}{\partial(V^*V)} + (v^*\dot{V} - \dot{V}^*v).$$

The computation of $\langle C|\hat{H}|C\rangle$, where $\hat{H}$ is given by (4) follows a similar path, and uses relation (8). The explicit expression of the Lagrangian (9) is finally obtained,

$$\begin{aligned}\mathcal{L} = \frac{\mathrm{i}}{2}&[(V^*\dot{V} - \dot{V}^*V)\mathcal{G}_n(V^*V) + (v^*\dot{V} - \dot{V}^*v)] \\ &- \omega(V^*V\mathcal{G}_n(V^*V) + v^*v) - G(V^*v + v^*V).\end{aligned} \tag{10}$$

We remark that $V^*V\mathcal{G}_n(V^*V) + v^*v$ is a constant of motion. The equation of motion for $V^*V\mathcal{G}_n(V^*V)$ is easily derived and is analogous to the previously obtained equation of motion for P. Indeed, let us write

$$v = \sqrt{\rho}\,\mathrm{e}^{\mathrm{i}\varphi} \qquad V = \sqrt{R}\,\mathrm{e}^{\mathrm{i}\Phi}.$$

The coherent state $|C\rangle$ is a function of the variables ρ, R, φ, Φ, t. It is convenient to introduce the notation

$$\langle C|\partial_x|C\rangle = \tfrac{1}{2}[\langle C|(\partial_x|C\rangle) - (\partial_x\langle C|)|C\rangle] \qquad x = \rho, R, \varphi, \Phi, t.$$

We find

$$\langle C|\partial_\Phi|C\rangle = R\mathcal{G}(R) \qquad \langle C|\partial_R|C\rangle = 0 \qquad \langle C|\partial_\varphi|C\rangle = \rho \qquad \langle C|\partial_\rho|C\rangle = 0.$$

In terms of the variables ρ, R, φ, Φ, the Lagrangian reads

$$\mathcal{L} = -(\dot{\Phi}R\mathcal{G}_n(R) + \dot{\varphi}\rho) - \omega(R\mathcal{G}(R) + \rho) - 2G\sqrt{R\rho}\cos(\varphi - \Phi).$$

The equations of motion read

$$\begin{aligned}&\partial_t(R\mathcal{G}_n(R)) - 2G\sqrt{R\rho}\sin(\varphi - \Phi) = 0 \\ &\partial_t\rho + 2G\sqrt{R\rho}\sin(\varphi - \Phi) = 0 \\ &-\dot{\Phi} - \omega - G\sqrt{\frac{\rho}{R}}\frac{\mathrm{d}R}{\mathrm{d}(R\mathcal{G}_n(R))}\cos(\varphi - \Phi) = 0 \\ &-\dot{\varphi} - \omega - G\sqrt{\frac{R}{\rho}}\cos(\varphi - \Phi) = 0.\end{aligned}$$

From the equations of motion it follows that $\rho + R\mathcal{G}_n(R)$ is conserved,

$$\partial_t[\rho + R\mathcal{G}_n(R)] = 0 \quad \Rightarrow \quad \rho + R\mathcal{G}_n(R) = \text{const} \tag{11}$$

and that

$$\dot{\Phi} - \dot{\varphi} - 2G\frac{\mathrm{d}}{\mathrm{d}\rho}\sqrt{R\rho}\cos(\Phi - \varphi) = 0$$

where we have used $\mathrm{d}R/\mathrm{d}(R\mathcal{G}_n(R)) = -\mathrm{d}R/\mathrm{d}\rho$ as follows from equation (11). Finally, we have

$$\partial_t^2\rho = 2G^2\frac{\mathrm{d}(R\rho)}{\mathrm{d}\rho} \tag{12}$$

analogously to the corresponding classical result (6).

The mean-field dynamical description of our model is sumarized by equation (12). By mean-field dynamics we mean a dynamical description based on the assumption that, to a good approximation, the time evolution of a coherent state wave packet proceeds along coherent state wave packets. The dynamical evolution of the quantal state is then transferred to the time dependence of the few parameters on which the coherent state depends. Mean-field dynamics deviates from the exact quantal dynamics in two respects: (1) in the mean-field approach, de-coherence effects are artificially suppressed; (2) quantum fluctuations associated with the conservation of constants of motion are also neglected. Of course, (2) is a consequence of (1). It is important to investigate when mean-field dynamics is valid and how reliable it is.

In our example, $\hat{P} = 2\hat{\sigma} = c^\dagger c + d^\dagger d$ is a constant of motion which is only conserved in the average by the coherent states considered. Quantum fluctuations associated with the conservation of $\hat{P}$ may be taken into account if modified coherent states which are themselves eigenstates of this operator are used to describe the dynamics. Coherent states of the form

$$|C\rangle_3 = \mathcal{N}_3 \exp[V\hat{d}^\dagger(F_n(\hat{N}))^{-1/2}\hat{c}]\hat{c}^{\dagger 2\sigma}|0\rangle \tag{13}$$

or

$$|C\rangle_4 = \mathcal{N}_4 \exp[V\hat{d}^\dagger\hat{c}]\hat{c}^{\dagger 2\sigma}|0\rangle$$

are eigenstates of $\hat{P}$. They belong to the eigenspaces of $\hat{P}$, which are subspaces spanned by the state vectors

$$|n_c, n_d\rangle \qquad n_c = 0, 1, \ldots, 2\sigma \qquad n_d = 0, 1, \ldots, 2\sigma \qquad n_c + n_d = 2\sigma.$$

It is particularly interesting to investigate the validity of the approximate dynamics constrained to these new types of states, since an improvement is expected with respect to the previously considered types. In the following, by coherent states 3, 4 (CSIII and CSIV) we mean, respectively, the coherent states $|C\rangle_3$, $|C\rangle_4$. The coherent state 4 is the $SU(2)$ coherent state [9].

5. Deformed bosons

Let us introduce the generalized deformed oscillator [3–5] which is defined in terms of the algebra generated by the operators $(1, \hat{d}', \hat{d}'^\dagger, \hat{N})$ and of the structure function

$$\Phi_n(x) \equiv [x] = F_n(x-1)x \qquad F_n(x) = \prod_{p=1}^{n-1}\left(1 + \frac{n}{p}x\right) \tag{14}$$

according with

$$\hat{d}'^\dagger = \hat{d}^\dagger\sqrt{F_n(\hat{N})} \qquad \hat{d}' = \sqrt{F_n(\hat{N})}\hat{d} \qquad \hat{N} = \Phi_n^{-1}(\hat{d}'^\dagger\hat{d}') \tag{15}$$

where $[\hat{d}, \hat{d}^\dagger] = 1$, $\hat{N} = \hat{d}^\dagger \hat{d}$. The function $F_n(x)$ is the same which was defined in section 2 by equation (2). Then we have

$$\hat{d}'^\dagger \hat{d}' = F_n(\hat{N} - 1)\hat{N} = [\hat{N}] \qquad \hat{d}' \hat{d}'^\dagger = F_n(\hat{N})(\hat{N} + 1) = [\hat{N} + 1]$$
$$[\hat{d}'^\dagger, \hat{N}] = -\hat{d}'^\dagger \qquad [\hat{d}', \hat{N}] = \hat{d}' \tag{16}$$

$$[\hat{d}', \hat{d}'^\dagger] = [\hat{N} + 1] - [\hat{N}]. \tag{17}$$

In terms of the deformed boson $\hat{d}'$ and the normal boson $\hat{c}$ the Hamiltonian reads

$$\hat{H} = \omega(\hat{c}^\dagger \hat{c} + \hat{N}) + g(\hat{d}'^\dagger \hat{c} + \hat{c}^\dagger \hat{d}').$$

The operator $2\hat{\sigma} = \hat{c}^\dagger \hat{c} + \hat{N}$ is a constant of motion, $[\hat{\sigma}, \hat{H}] = 0$.

We now introduce the coherent state

$$|v, V\rangle = \mathcal{N} \exp_\Phi(V \hat{d}'^\dagger) \exp(v \hat{c}^\dagger)|0\rangle \tag{18}$$

where the deformed exponential is defined as

$$\exp_\Phi(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]!}$$

where $[k]! = [k][k-1]\cdots[1]$, and $[x]$ is defined as in equation (14). It is clear that $|0\rangle$ satisfies $\hat{d}'|0\rangle = \hat{c}|0\rangle = 0$ and $|v, V\rangle$ is an eigenstate of the annihilation operator $\hat{d}'$,

$$\hat{d}'|v, V\rangle = V|v, V\rangle.$$

The coherent state (18) is just a different way of writing the coherent state (7). Then we have for $\mathcal{H} = \langle v, V|\hat{H}|v, V\rangle$

$$\mathcal{H} = \omega(V^* V \mathcal{G}_n(V^* V) + v^* v) + G(V^* v + v^* V)$$

which corresponds to the last two terms of (10).

6. Time evolution

The time evolution of the system is determined from the action principle [11]

$$\delta \int \langle C|\mathrm{i}\hbar\partial_t - \hat{H}|C\rangle \mathrm{d}t = 0. \tag{19}$$

If $|C\rangle$ is free from any constraint, equation (19) leads to the exact time evolution. If not, equation (19) leads to the 'best' time evolution compatible with the imposed constraint. Here, and in the following, $|C\rangle$ is any of the coherent states introduced above. For CSII, it is also instructive to consider the equations of motion obtained when the pairs of variables (Φ, R) and (φ, ρ) are replaced by the pairs of canonical variables (ϕ, σ) and (χ, ν), defined in terms of the old variables (V, V^*), (v, v^*) by

$$V = \mathrm{i}\sqrt{R}\, \exp(-\mathrm{i}\chi)\exp(-\mathrm{i}\phi/2) \qquad v = \sqrt{\rho}\exp(-\mathrm{i}\phi/2)$$

and

$$\sigma = \tfrac{1}{2}\langle C|\hat{c}^\dagger \hat{c} + \hat{d}^\dagger \hat{d}|C\rangle = R\mathcal{G}(R) + \rho \qquad \nu = R\mathcal{G}(R).$$

We obtain

$$\langle C|\mathrm{i}\partial_\phi|C\rangle = \sigma \qquad \langle C|\mathrm{i}\partial_\sigma|C\rangle = 0 \qquad \langle C|\mathrm{i}\partial_\chi|C\rangle = \nu \qquad \langle C|\mathrm{i}\partial_\nu|C\rangle = 0.$$

In terms of the new variables we have

$$\mathcal{H} = \langle C|H|C\rangle = 2\omega\sigma + 2G\sqrt{\frac{\nu}{L(\nu)}}\sqrt{2\sigma - \nu}\,\sin\chi$$

where $L(\nu) = \mathcal{G}[R(\nu)]$. From the equations of motion we get

$$\dot{\nu} = -2G\sqrt{\frac{\nu}{L(\nu)}}\sqrt{2\sigma - \nu}\cos\chi\,.$$

Using the total energy E to eliminate χ we obtain the equation which determines the time evolution of the system in the present approach

$$\dot{\nu}^2 = 4G^2\frac{\nu}{L(\nu)}(2\sigma - \nu) - (E - 2\omega\sigma)^2. \tag{20}$$

For the coherent states CSIII and IV, which preserve the number of particles, we introduce the new pair of canonical variables (ν, χ) such that $V = \mathrm{i}\sqrt{R(\nu)}\,\exp(-\mathrm{i}\chi)$ and get for the energy and equation of motion for ν, respectively,

$$H = 2\omega\sigma + 2\sigma G\sqrt{R}\mathcal{F}_i(\sqrt{R})\sin\chi \qquad \mathrm{i} = \mathrm{III},\ \mathrm{IV}$$

and

$$\dot{\nu}^2 = 16G^2\sigma^2\mathcal{F}_i^2(\sqrt{R}) - (E - 2\omega\sigma)^2$$

where

$$\mathcal{F}_{\mathrm{III}}(\sqrt{R}) = \sum_{n=0}^{2\sigma-1} R^n\frac{(2\sigma-1)!}{n!(2\sigma-1-n)![n-1]!} \qquad \mathcal{F}_{\mathrm{IV}}(\sqrt{R}) = \sum_{n=0}^{2\sigma-1} R^n\frac{(2\sigma-1)!\sqrt{F(n)}}{n!(2\sigma-1-n)!}.$$

7. Numerical results and conclusions

The reliability of the time evolution predicted by coherent states has been investigated numerically in a two-boson schematic model by comparing the coherent state dynamics with the exact one.

Recalling that $\hat{\sigma}$ is a constant of motion, we have considered systems characterized by different values of $\langle 2\hat{\sigma}\rangle = \langle\hat{c}^\dagger\hat{c}\rangle + \langle\hat{d}^\dagger\hat{d}\rangle = 2, 10, 20$. The results for the phonon multiplicities $n = 2$ and $n = 4$ are given, respectively, in figures 1 and 2. Time is measured in units of (g^{-1}). For $t = 0$ the system is described by an appropriate coherent state with $\langle\hat{c}^\dagger\hat{c}\rangle = 2\sigma$ and $\langle\hat{d}^\dagger\hat{d}\rangle = 0$. This is the initial condition. The exact time evolution, which is obtained when the coherent state constraint is relaxed for $t > 0$, is represented by the solid lines. The dotted, small-dashed, long-dashed and dash-dotted lines represent the constrained time evolutions for the coherent states CSI, CSII, CSIII and CSIV, respectively. The exact results exhibit a noticeable loss of coherence, especially for larger 2σ values. We observe that there are no instants for which the exact time evolution of the expectation value $\langle\hat{c}^\dagger\hat{c}\rangle$ vanishes and for $n = 2$ it does not even recover the initial value $2\sigma = 10$ or 20 . This behaviour is apparently in strong contrast with the constrained time evolution for coherent states. Contrary to the exact time evolution, the coherent state dynamics leads to an oscillatory behaviour between the initial value 2σ and zero. In this case, $\langle\hat{c}^\dagger\hat{c}\rangle$ vanishes periodically. However, the length of time spent close to zero is much shorter than the length of time spent close to the maximum. The difference between the exact and constrained dynamics is not too pronounced and the time averages of the corresponding time evolutions are in qualitative agreement, except for the CSI coherent state. In table 1 the time average of the number of bosons c is given for coherent states CSII, CSIII, CSIV. For the exact value we have taken the average over several

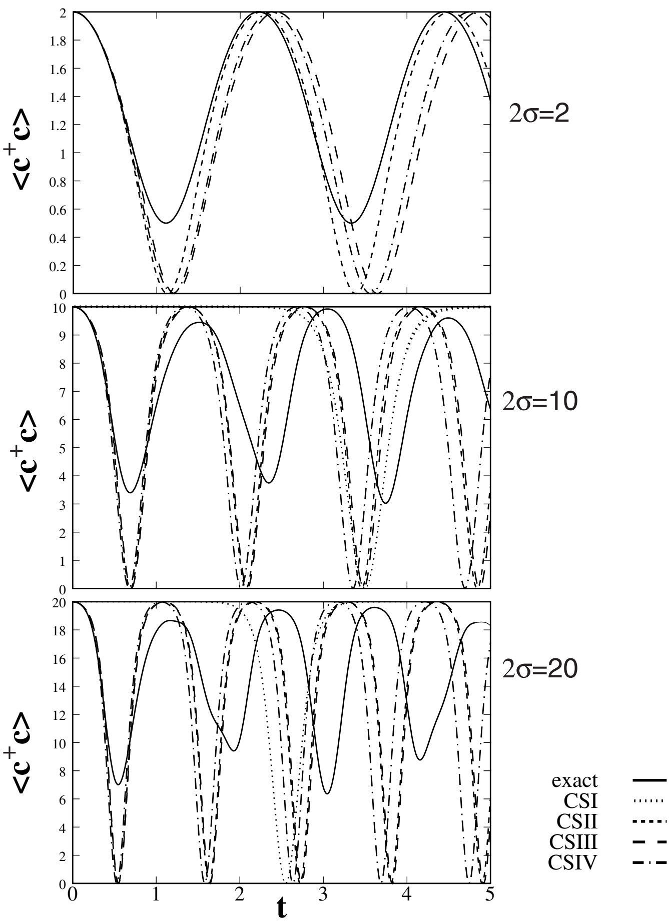

Figure 1. Time evolution of the average number of bosons $\hat{c}$ for $n = 2$ and $2\sigma = 2, 10, 20$. The initial condition is $\langle\hat{c}^\dagger\hat{c}\rangle = 2\sigma$ and $\langle\hat{d}^\dagger\hat{d}\rangle = 0$.

Table 1. Time-averaged value of $\langle c^\dagger c\rangle$.

	$n = 2$			$n = 4$		
2σ	2	10	20	2	10	20
CSII	1.18	6.81	14.23	1.50	9.22	19.10
CSIII	1.15	6.79	14.21	1.44	9.19	19.08
CSIV	1.14	6.70	14.07	1.42	9.09	18.93
Exact	1.31	7.16	14.88	1.91	9.82	19.79

periods, since it has some small fluctuations. The period of the revivals is given in table 2. Except for $2\sigma = 2$, the period of the revivals is shorter than that obtained with the coherent states. For $n = 4$, CSII and CSIII give a period which is almost twice as large as the exact value. The performance of the CSI state is not acceptable: it neither describes the period nor the behaviour at instants for which $\langle\hat{c}^\dagger\hat{c}\rangle$ is close to its maximum value. We only show it in the figures for $2\sigma = 10, 20$ and $n = 2$ (dotted line). Close to the instant for which $\langle\hat{c}^\dagger\hat{c}\rangle$ attains its maximum value, the remaining three coherent states (CSII, CSIII, CSIV) behave

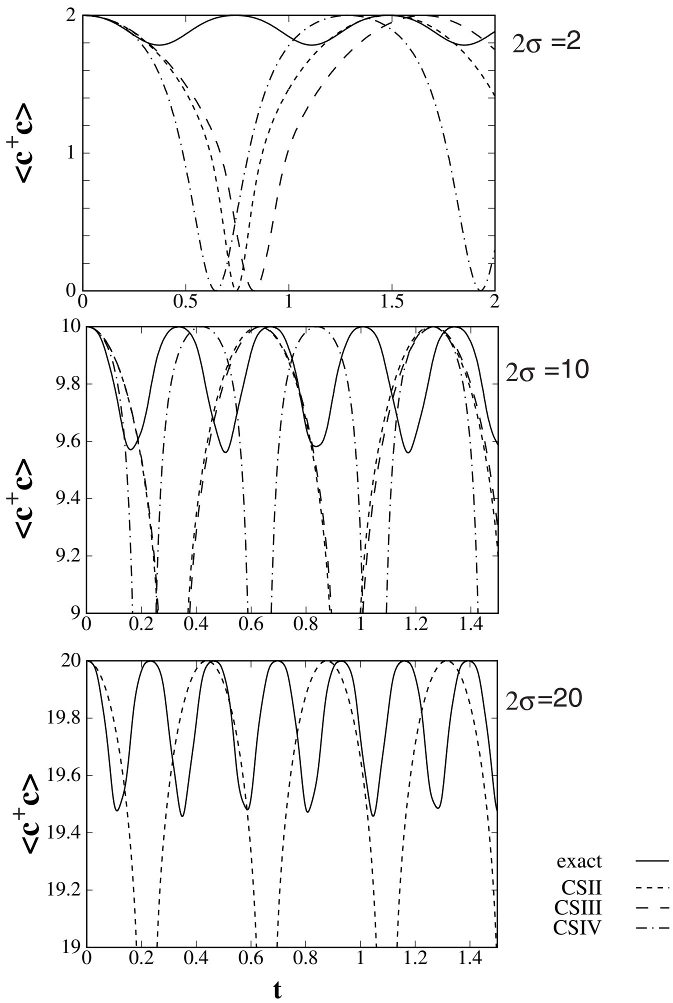

Figure 2. Time evolution of the number of bosons for $n = 4$ and $2\sigma = 2$, 10, 20.

Table 2. Period T for $n = 2$ and $n = 4$ calculated with different coherent states and particle numbers.

2σ	2	10	20
	$n = 2$		
CSII	2.257	1.380	1.087
CSIII	2.425	1.390	1.095
CSIV	2.370	1.340	1.055
Exact	2.222	1.522	1.221
	$n = 4$		
CSII	1.489	0.628	0.438
CSIII	1.662	0.634	0.432
CSIV	1.285	0.42	0.285
Exact ($2T$)	1.48	0.670	0.46

very similarly, describing reasonably well the decay of the initial state. For $2\sigma = 10, 20$ coherent states CSII and III are equivalent. For $n = 4$ and $2\sigma = 20$ we only represent CSII, because, as can be seen from tables 1 and 2, both the time average and period are almost equal. The performonce of CSIV is not as good as the one of CSII and CSIII. In general, the generalized coherent states which depend nonlineraly on the particle number behave better.

The coherent state CSII only conserves in the average the constant of motion $\hat{\sigma}$. In spite of this fact, its performance is not inferior to the other coherent states. In fact for $2\sigma = 2$, when the differences between CSII and CSIII are greater, it seems to work better than CSIII. However, we have noted that CSII is giving poor results when static properties, such as the ground-state energy are calculated [13].

It would be nice to understand when does the coherent state description break down, how to implement de-coherence effects, and why, for $n = 4$, the coherent state description leads to a period of oscillations which is twice the true one.

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